Classification of rotational figures of equilibrium

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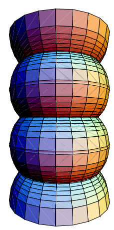
I. Introduction

In the 1840's, blind Belgian physicist and mathematician Joseph Plateau conducted experiments with rotating liquid drops. He intended his centimeter sized drops held together by surface tension to be models for immense celestial liquid masses held together by self gravitation. While this interpretation was later shown to be inaccurate, determining the shape and stability of rotating liquid drops has retained importance as a source of numerical and mathematical challenges and in applications to nuclear physics.

Among Plateau's accomplishments was his observation of toroidal-shaped drops. At the time, he challenged mathematicians to prove the existence of formal solutions to the governing equations leading to toroidal shapes.

Robert Gulliver proved in 1984 the existence of rotationally symmetric tori. He also showed that all possible solutions lie in a two parameter family represented by the (λ,c)-plane on the right.

Our work seeks to classify all solutions in this family. With intuition gained from numerical calculations, we can prove existence of toroidal solutions for all λ.



Nodoid Type

II. Model Equation

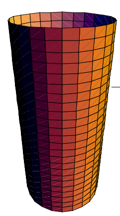
The energy of a rotating drop enclosing volume *V* is given by

$$E = \sigma \,|\, S \,| - \tfrac{1}{2}\rho\omega^2 \int_V r^2$$

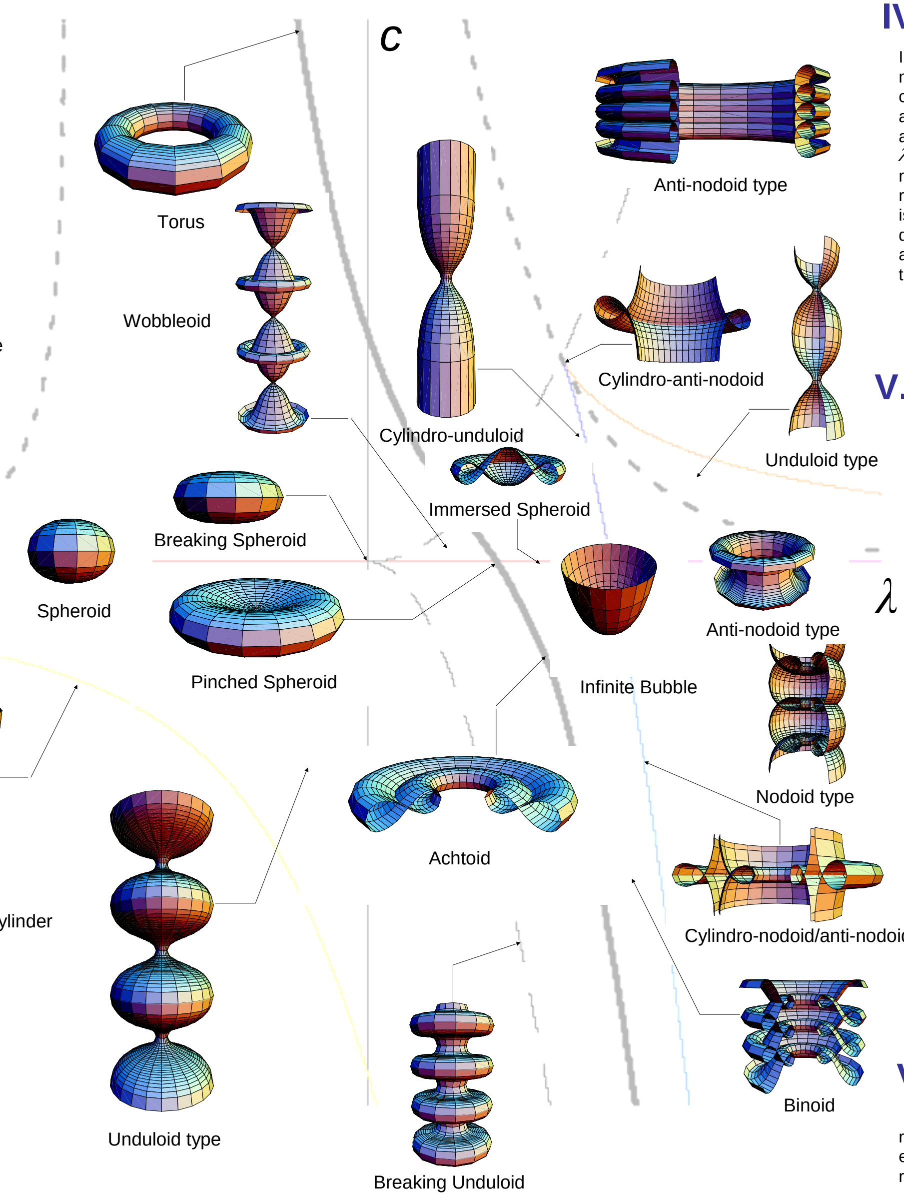
where *S* is the free surface, σ is the surface tension, ρ is the density, and ω is the angular velocity.

Representing the drop's meridian curve as a graph *u=u(r)* of the radial distance from the axis, the *Euler-Lagrange* equation for the equilibrium configuration is

$$\frac{1}{r}\left(\frac{ru'}{\sqrt{1+u'^2}}\right)' = -4ar^2 + 2\lambda$$



Cylinder

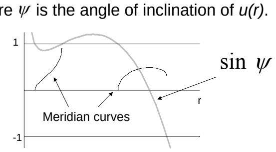


III. Classification Method

By scaling, we may assume a =1 and perform an initial classification through analysis of

$$\sin\psi = \frac{u'}{\sqrt{1+u'^2}} = -r^3 + \lambda r + \frac{c}{r}$$

where ψ is the angle of inclination of *u(r)*.



We refine our classification by considering

$$h = u(r_2) - u(r_1) = \int_{r_1}^{r_2} \frac{\sin\psi \, dt}{\sqrt{1-\sin\psi^2}}$$

which is the height difference in the endpoints of a half period of a (periodic) meridian curve.

"I think it very probable that if calculation could approach the general solution of this great problem, and lead directly to the determination of all the possible figures of equilibrium, the annular figure would be included among them."

J. Plateau

IV. Physical Drops

In order to compare a calculated meridian curve of a simply connected drop (c=0) with an actual physical drop, the appropriate Lagrange parameter λ must be determined. The relationship between the rotation rate and the Lagrange parameter is complicated but may be determined numerically and analyzed via the implicit function theorem in certain special cases.

V. Toroidal Figures

In 1984, R. Gulliver proved:

(1) For each c $\geq$ 3/16, there is a λ(c) for which the corresponding figure is a torus with convex cross section.

(2) There is an interval c $\in (0,\varepsilon)$ and a smooth function λ(c) for which the corresponding figureis a torus.

We prove the existence of toroidal solutions for all c and the figure exhibits numerical calculations that suggest that associated with each c there is a single toroidal solution.

We show in addition that there are no toroidal solutions outside specified regions and give a rigorous classification of solutions in these regions.

(R. Gulliver, Tori of prescribed mean curvature and the rotating drop. *Soc. Math. de France*)

VI. Discussion

Our results may be used to model rotating liquid drops in low gravity environments with or without contacting rigid support structures.

The uniqueness of toroidal solutions and a fuller understanding of the dependence of the Lagrange parameter are subjects of ongoing interest.